EFFECT OF HEISENBERG'S PRINCIPLE ON CHANNEL CAPACITY*

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The limitations imposed by thermodynamics on the amount of energy necessary to transmit one bit of information, has been discussed by Felker and Pierce.[1] A minimum of $kT \log_e 2$ ergs per bit was obtained. k is Boltzmann's constant, and T is absolute temperature.

Professor Fano has suggested that Heisenberg's principle may affect channel capacity.

The following analysis shows that the energy necessary to transmit one bit is not appreciably increased by quantum mechanical considerations, providing that

$$w \ll \frac{2}{3\pi} \frac{kT}{h}$$

w is the channel bandwidth in cycles per second and h is Planck's constant.

Consider a simple channel of bandwidth w, with a signal power per cycle of s/w ergs, and a noise power per cycle of N/w ergs. In order to optimize efficiency, we shall assume $N \gg s$.[1]

A single measurement of the signal by the receiver will, according to quantum mechanics, involve some uncertainty in its energy. Let this uncertainty in energy be denoted by ϵ. It will contribute the equivalent of less than 2ϵ ergs of additional noise power per cycle.

Associated with this energy uncertainty is an uncertainty of time of measurement, which we will call τ. Heisenberg's principle states that

$$\tau = \frac{h}{\epsilon}$$

is about the most accurate in time–of–measurement we can obtain.

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[1] J.H. Felker, "A Link Between Information and Energy," Proc. I.R.E., Vol. 40, pp. 728–729, June, 1952

To find the power of the noise equivalent to the time–of–measurement uncertainty, consider that one is observing an equivalent of signal–plus–noise of flat spectrum

$$G(f) = \frac{s}{w} + \frac{N}{w} + 2\epsilon \text{ ergs}$$

extending from $f = 0$ to $f = w$.

Measuring the signal at time $t-r$, and using this measurement as an estimate of the value of the signal at time t, will almost always result in some error. The size of this error is the same as the amplitude of the difference between the original signal and the output of a hypothetical delay circuit of delay time t, into which the signal could be fed.

The delay circuit is of frequency response

$$\epsilon^{2\pi i \tau f}$$

The error signal can be obtained from the original signal, by subjecting it to a filter of frequency response

$$1 - \epsilon^{2\pi i \tau f}$$

The spectrum of the error will be

$$\left(\frac{s}{w} + \frac{N}{w} + 2\epsilon\right) \left|1 - \epsilon^{2\pi i \tau f}\right|^2$$

To find the error power per cycle, we integrate this spectrum over all frequencies at which the signal exists, and divide by w, obtaining

$$\frac{1}{w} \int_0^w \left|1 - \epsilon^{2\pi i \tau f}\right|^2 \left(\frac{s}{w} + \frac{N}{w} + 2\epsilon\right) df$$

additional ergs of "noise" power per cycle.

Although the error in time of measurement will not always be τ, but will have a probability distribution of zero mean and width τ, a more exact treatment results only in an unimportant scale factor of the order of unity.

We may approximate this integral rather well by

$$\frac{4\pi^2}{3} N\tau^2$$

if

$$s \ll N\,, \quad 2\pi w\tau \ll 1$$

and

$$2\epsilon \ll \frac{N}{w}$$

The total additional noise contribution due to both energy and time uncertainties is

$$\frac{4\pi^2}{3} Nw\tau^2 + 2\epsilon = \frac{4\pi^2}{3} Nw \frac{h^2}{\epsilon^2} + 2\epsilon$$

Since we may make ϵ arbitrary, let us choose it so that this total additional noise is minimized. We obtain

$$\epsilon = \left(\frac{4\pi^2}{3} Nwh^2 \right)^{1/3} \text{ ergs}$$

The total equivalent increase in noise power then becomes

$$3 \left(\frac{4}{3} \pi^2 Nwh^2 \right)^{1/3} \text{ ergs}$$

If this noise is to contribute negligibly to the channel equivocation, it must be much less than N/w, the ordinary thermal noise power per cycle, that is

$$3 \left(\frac{4}{3} \pi^2 Nwh^2 \right)^{1/3} \ll \frac{N}{w}$$

or

$$6\pi wh \ll \frac{N}{w}$$

Since $N/w = 4kT$[1], we obtain

$$w \ll \frac{2}{3\pi} \frac{kT}{h}$$

From purely dimensional considerations it can also be shown that additional channel equivocation approaches zero, as wh/kT approaches zero, but no clear indication could be obtained as to the relative rates of approach.

Using $T = 300$ degrees absolute, we find $w \ll 1.6 \times 10^{13}$ cycles per second at room temperature.

This limitation on bandwidth is not serious from a practical standpoint, but even if one did want to transmit information faster than this, it would be possible to use several independent channels in parallel, keeping the bandwidth of each below the limit, and still obtain an over–all channel capacity in excess of that suggested by the formula.

From the foregoing, it appears that Heisenberg's principle imposes no additional efficiency limitations on information channels.